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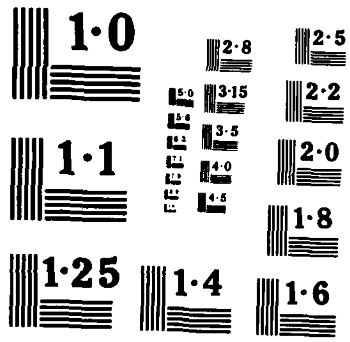

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A FUNCTIONAL EQUATION GOVERNING MOVING PHASE BOUNDARIES IN AN ELASTIC BAR

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ABSTRACT

Certain elastic solids when subjected to sufficiently high loads abruptly change their mechanical properties and yet continue to respond elastically to further loading. In one dimension such mechanically induced elastic phase transitions may be due to a non-monotonic stress-strain relationship. This appears to be particularly true for certain mineral crystals, such as calcite.

This work considers a one-dimensional dynamical problem for a special material. The problem reduces to determining the location of the internal moving boundary separating distinct elastic phases. This phase boundary is similar to a gas dynamical shock wave. For the problem considered here, this phase boundary is shown to be governed by a functional equation of the form

$$\phi(\phi(t)) + F(\phi(t)) + t = 0$$

for the unknown $\phi(t)$, where $F()$ is a known function involving the boundary conditions. The unusual equation is derived by considering the effect of acoustic waves repeatedly reflecting between the phase boundary and the external boundary. The equation is shown to possess a unique solution and is treated asymptotically to determine the large-time behavior of the phase boundary.

AMS (MOS) Subject Classifications: 35L65, 35L67, 39B05, 41A60, 73D05

Key Words: phase transitions, elastic solids, Functional equations

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SIGNIFICANCE AND EXPLANATION

Certain elastic solids when subjected to sufficiently high loads abruptly change their mechanical properties and yet continue to respond elastically to further loading. In one dimension such mechanically induced elastic phase transitions may be due to a non-monotonic stress-strain curve. This work investigates the cumulative reflection of acoustic waves between the external boundary of the solid and the internal moving boundary separating distinct elastic phases. This latter phase boundary is similar to a gas dynamical shock wave. For the material introduced in this work, a functional equation governing the trajectory of a phase boundary is derived and shown to have a unique solution. This equation is treated asymptotically to determine the large time behavior of the phase boundary.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the author of this report.

A FUNCTIONAL EQUATION GOVERNING MOVING PHASE BOUNDARIES IN AN ELASTIC BAR

Thomas J. Pence

1. Introduction

Certain elastic solids when subjected to sufficiently high loads will abruptly change their mechanical properties and yet continue to respond elastically to further loading. An example of this type of behavior is found in the mineral calcite, which suffers a change in its crystal structure whenever it is sufficiently compressed [1]. James [2], [3] cites other examples in materials ranging from metals to natural rubbers and various polymers. The possibility that a material may cease to behave in an expected manner has an impact on a host of geophysical and engineering problems. It is also interesting to note that metals which undergo a phase transition due to a change in temperature are being promoted as a means for performing specific engineering tasks [4].

The study of such phenomena within the theory of elasticity centers around material models involving non-convex strain energies (in addition to [2], [3] see Ericksen [5], Knowles & Sternberg [6], [7], [8], Abeyaratne [9], [10] and Pence [11]). In one dimension, these models are equivalent to theories of materials with a non-monotonic stress-strain relation. This type of model is not new; the theory of a Van der Waals fluid is based upon a non-monotonic constitutive relation. Indeed, phase boundaries in certain solids may be kinematically similar to the liquid-vapor phase boundary in a Van der Waals fluid. Admissibility criteria for the later type of phase boundary have been studied by Serrin [12], Slemrod [13], and Slemrod & Hagan [14].

This paper considers the problem of determining the motion of the front associated with a phase transition for a model elastic material in one dimension. We imagine a bar in which transverse displacements are absent and we let $u(x,t)$ be the longitudinal displacement in a Lagrangian frame. We also let σ, $\epsilon = \frac{\partial u}{\partial x}$ and $v = \frac{\partial u}{\partial t}$ denote

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respectively the stress, strain and velocity in the bar. The material is assumed to obey a stress-strain relation like that depicted in figure 1.

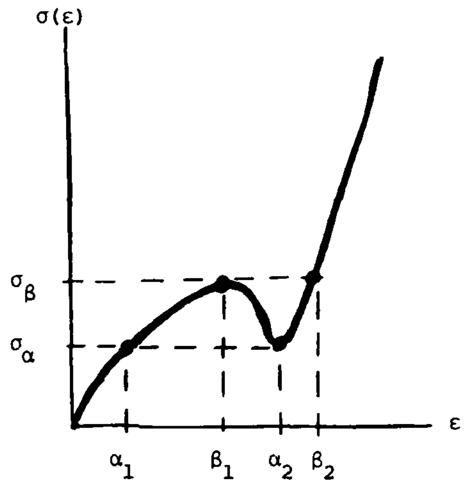

Figure 1.

The function $\sigma(\varepsilon)$ is taken to be of unlimited smoothness, concave down on the first ascending branch, and is assumed to approach ∞ as $\varepsilon \to \infty$. These restrictions are not essential to the general theory; what is essential is the existence of two distinct ascending branches. It is these two branches which are associated with distinct material phases. We consider a tensile load $g(t)$ applied at $x = 0$ to a bar which is initially undeformed and at rest, and which is also taken to be semi-infinite in order to avoid having to treat waves reflected from a second fixed boundary. The equation of motion is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} . \tag{1.1}$$

It is to be solved subject to

$$\sigma\big(\varepsilon(0,t)\big) = g(t), \qquad (t > 0), \tag{1.2}$$

$$\varepsilon(x,0) = 0, \quad v(x,0) = 0, \quad (x > 0) . \tag{1.3}$$

As discussed in [11], solutions will include discontinuities in σ, ε, v in the event that $g(t) > \sigma_\beta$. If $x = s(t)$ is the trajectory in the (x,t)-plane of such a discontinuity, the jump in the dynamical fields across $x = s(t)$ are to be restricted by the familiar conditions

$$\frac{ds}{dt} [\![\varepsilon]\!] + [\![v]\!] = 0, \quad \frac{ds}{dt} [\![v]\!] + [\![\sigma]\!] = 0 . \tag{1.4}$$

Phase boundaries are those particular discontinuity fronts that separate strains associated with the two different ascending branches of fig. 1.

The model is explained more fully in section 2, where we also consider the special problem of an impulsive load ($g(t)$ a step function). This problem is studied in order to touch upon some questions concerning uniqueness as well as in order to discuss an unusual - and physically desirable - feature of certain mathematical solutions: phase boundaries which travel at lower speeds than the sound speeds of the material in both adjacent phases. In section 3, we return to a more general loading program, and introduce the notion of an energy confining phase boundary. In section 4 we restrict attention to materials for which the second ascending branch is linear. This gives rise to a problem involving a linear wave equation in a region of the (x,t)-plane which is partially bounded by the (as yet unknown) location of a phase boundary. This problem is reducible in certain cases to a functional equation of the form

$$\phi\big(\phi(z)\big) - 2\phi(z) + z = Q\big(\phi(z)\big) , \tag{1.5}$$

where ϕ, the function to be determined, is subject to some additional conditions and Q is a known monotonically increasing function which incorporates the function $g(t)$. The unusual equation (1.5) is derived by considering the cumulative effect of acoustic waves repeatedly reflecting between the moving phase boundary and the fixed external boundary. In section 5, we prove the existence of a unique solution ϕ by means of an iterative scheme. A more delicate argument subsequently guarantees the differentiability of this solution. In the final section, eqn. (1.5) is treated asymptotically to determine the large-time behavior of the phase boundary. The resulting asymptotic solution agrees

with the particular solution of the impulsive load problem presented in section 2 in the limit $t \to \infty$, x/t fixed.

Equations in which a function to be determined appears in its own argument arise in other branches of mathematics. Specifically, (1.5) is a special case of the equation of invariant curves. Thus the theoretical results presented in section 5 extend some previous results in that theory (see Chapter XIV of [15]). In the field of continuum mechanics, problems involving wave propagation can lead to equations involving a delay in their argument. Although this work is devoted to solid mechanics, a set of functional equations have been considered by Seymour & Mortel [16] in connection with the study of oscillations occurring in an inviscid gas confined in a closed tube. In fact, their eqn. 1. can be manipulated into the form of (1.5), although the properties of Q (H in [16]) differ. In both this work and [16], the appearance of a function in its own argument is due to the unknown time delay for the return of a reflected signal. It is intriguing that in [16] this delay is due to an amplitude dependent acoustic speed, whereas in this work the sound speeds are known prior to the derivation of (1.5). In the problem considered here, the unknown delay stems from the unknown location of the reflecting phase boundary.

2. Solutions for the Impulsive Load Problem

The sound speed of the material is given by the value $\sqrt{\sigma'(\varepsilon)}$, where the ' symbol is the usual notation for derivative. Shock waves arise naturally in the solution of (1.1-4) due to the intersection of characteristic curves in the (x,t)-plane. In contrast, phase boundaries occur in this theory because no single branch of the $\sigma(\varepsilon)$-curve can accommodate stress values both less than σ_α and greater than σ_β. As discussed in [3], phase boundaries differ from the aforementioned shock waves in that they may travel at a much lower velocity than the sound speeds of the material in each adjacent phase.

In an equilibrium setting, configurations involving strains on the *descending branch* of the $\sigma(\varepsilon)$-curve are extremely unstable [2]. For the dynamical problem, the sound speeds are no longer real on this branch. This prompts us to seek solutions to (1.1-4) which avoid this branch altogether. Thus every time-interval during which $g(t)$ traverses the range $[\sigma_\alpha,\sigma_\beta]$ must generate at least one jump in $\varepsilon(0,t)$ between the two ascending branches. Let us suppose that $g(t)$ increases monotonically from 0 to a value greater than σ_β and discuss solutions with only one such jump, say first occurring at $t = t^*$. Then $\sigma_\alpha < g(t^*) \equiv \sigma_J < \sigma_\beta$ where σ_J is the stress at which the change of phase first occurs.

A criterion for selecting σ_J remains elusive. A well-known stability argument yields the value associated with the Maxwell line [3]. Recent work in this vein has addressed how strain rates and strain gradients can affect σ_J (see [12], [13], [14] for the corresponding problem in a Van der Waals gas), as well as how σ_J may be affected by a phase boundary surface energy [17]. This paper addresses the simplier and more immediate problem of constructing solutions to (1.1-4) for a given value of σ_J. Thus we append the following restrictions upon the strain at $x = 0$:

$$0 < \varepsilon(0,t) < \varepsilon_J \quad \text{or} \quad \varepsilon(0,t) > \bar{\varepsilon}_J, \quad (t > 0)\ , \tag{2.1}$$

where ε_J and $\bar{\varepsilon}_J$ denote respectively the first - and second - phase values of strain associated with σ_J via

$$\sigma(\varepsilon_J) = \sigma(\bar{\varepsilon}_J) = \sigma_J, \quad \alpha_1 < \varepsilon_J < \beta_1, \quad \alpha_2 < \bar{\varepsilon}_J < \beta_2 . \tag{2.2}$$

One might expect condition (2.1) to render solutions of (1.1-4) unique, this, however, is not necessarily true. Consider first the impulsive-load problem

$$g(t) = \begin{cases} 0, & t = 0 , \\ \sigma_\infty > \sigma_J, & t > 0 . \end{cases} \tag{2.3}$$

Following James [3], we seek a similarity solution of the form

$$\varepsilon(x,t) = \tilde{\varepsilon}(\lambda), \quad v(x,t) = \tilde{v}(\lambda), \quad \lambda = x/t , \tag{2.4}$$

where $\tilde{\varepsilon}(\cdot)$, $\tilde{v}(\cdot)$ are piecewise differentiable functions defined on $0 < \lambda < \infty$. If either function is discontinuous at $\lambda = \lambda_i$, then $x = \lambda_i t$ is either a shock or phase boundary. Since every ray $x = \lambda t$ meets at the origin, we shall in this section interpret (2.1) as requiring $\tilde{\varepsilon}(\lambda) \in [0,\varepsilon_J] \cup [\bar{\varepsilon}_J,\infty]$. Equations for $\tilde{\varepsilon}$ and $\tilde{v}$ follow by introducing (2.4) into (1.1). One finds that $\tilde{\varepsilon}(\lambda)$ is either constant or obeys

$$\sigma'\big(\tilde{\varepsilon}(\lambda)\big) = \lambda^2 . \tag{2.5}$$

The function $\tilde{v}(\lambda)$ satisfies

$$\frac{\partial \tilde{v}}{\partial \lambda} = -\lambda \frac{\partial \tilde{\varepsilon}}{\partial \lambda} . \tag{2.6}$$

This last equation, in conjunction with (1.3-4) indicate that $\tilde{v}$ follows from $\tilde{\varepsilon}$ as

$$\tilde{v}(\lambda) = \begin{cases} -\displaystyle\int_0^{\tilde{\varepsilon}(\lambda)} \sqrt{\sigma'(s)}\, ds, & \lambda_1 < \lambda < \infty , \\ \tilde{v}(\lambda_i-) - \displaystyle\int_{\varepsilon(\lambda_i)}^{\tilde{\varepsilon}(\lambda)} \sqrt{\sigma'(s)}\, ds, & \lambda_{n+1} < \lambda < \lambda_n \quad (n = 1, \ldots, N - 1) , \\ & 0 < \lambda < \lambda_N , \end{cases} \tag{2.7}$$

where $\lambda_1 > \lambda_2 > \ldots > \lambda_N$ are the values of λ at which $\tilde{\varepsilon}$ is discontinuous and

$$\tilde{v}(\lambda_i-) \equiv \tilde{v}(\lambda_i+) + \lambda_i \left[\tilde{\varepsilon}(\lambda_i+) - \tilde{\varepsilon}(\lambda_i-)\right] . \tag{2.8}$$

Thus it merely remains to find an $\tilde{\varepsilon}(\lambda)$ which:

(a) is either constant or obeys (2.5) on some partitioning of $0 < \lambda < \infty$ into open intervals,

(b) yields the correct boundary values for $\varepsilon(x,t)$ via the relations,

$$\tilde{\varepsilon}(\infty) = 0, \qquad \tilde{\varepsilon}(0) = \varepsilon_\infty \quad \text{where} \quad \sigma(\varepsilon_\infty) = \sigma_\infty \,, \tag{2.9}$$

(c) at points of discontinuity λ_i obeys

$$\lambda_i^2 \left[\tilde{\varepsilon}(\lambda_i) \right] = \left[\sigma\left(\varepsilon(\lambda_i)\right) \right] \,, \tag{2.10}$$

and

(d) has range $[0,\varepsilon_J] \cup [\bar{\varepsilon}_J,\infty)$.

Notice that (2.8), (2.10) are necessary and sufficient to guarantee the discontinuity conditions (1.4). We now present a method for generating solutions $\tilde{\varepsilon}(\lambda)$ obeying (a) - (d). The constriction makes use of an upper concave envelope of portions of the function $\sigma(\varepsilon)$.

A function $h(x)$ defined on $a \leq x \leq b$ is concave if $h\left(\lambda x + (1 - \lambda)y\right) \geq \lambda h(x) + (1 - \lambda)h(y)$ for all $x,y \in [a,b]$ and all $\lambda \in [0,1]$. The upper concave envelope $\hat{k}(x)$ of a function $k(x)$ is the smallest concave function obeying $\hat{k}(x) \geq k(x)$. Thus at each fixed x_0 either $\hat{k}(x_0) = k(x_0)$ or $\left(x_0, \hat{k}(x_0)\right)$ lies on a line segment bridging two points on the graph of k. Figure 2 depicts $\hat{\sigma}(\varepsilon)$, the upper concave envelope of $\sigma(\varepsilon)$ in the domain $[0,\varepsilon_\infty]$ for a value ε_∞ as given in (2.9).

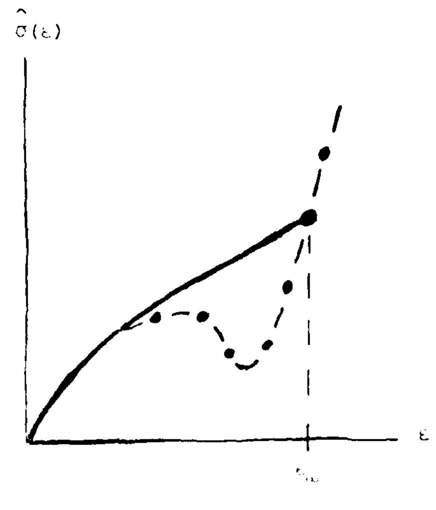

Figure 2. Upper concave envelope of $\sigma(\varepsilon)$ given in figure 1, for a specific value ε_∞.

Since the derivative $\hat{\sigma}'(\varepsilon)$ decreases continuously from $\hat{\sigma}'(0)$ to $\hat{\sigma}'(\varepsilon_\infty)$ the equation

$$\lambda^2 = \hat{\sigma}'(\tilde{\varepsilon}) , \tag{2.11}$$

has at least one solution $\tilde{\varepsilon}$ for each value $\lambda \in \left[\sqrt{\hat{\sigma}'(\varepsilon_\infty)}, \sqrt{\hat{\sigma}'(0)}\right]$. The value $\tilde{\varepsilon}$ is unique if $\tilde{\varepsilon}$ does not lie on a linear portion of $\hat{\sigma}(\varepsilon)$. Let λ_i be the positive square root of the slope of the $i^{\underline{th}}$ linear portion of $\hat{\sigma}(\varepsilon)$. Then $\lambda_1 > \lambda_2 > \ldots > \lambda_N$ and (2.11) has an interval of solutions $\tilde{\varepsilon}$ for each $\lambda = \lambda_i$. Thus (2.11) implicitly defines a decreasing function $\tilde{\varepsilon}(\lambda)$ on $\sqrt{\hat{\sigma}'(\varepsilon_\infty)} < \lambda < \sqrt{\hat{\sigma}'(0)}$ which is discontinuous at each $\lambda = \lambda_i$. By construction $\tilde{\varepsilon}$ obeys (2.5) between these discontinuities. At the same time, (2.10) holds for all $\lambda = \lambda_i$ on account of λ_i^2 being the slope of a secant line to the curve $\sigma(\varepsilon)$. Since $\tilde{\varepsilon}\left(\sqrt{\hat{\sigma}'(\varepsilon_\infty)}\right) = \varepsilon_\infty$ and $\tilde{\varepsilon}\left(\sqrt{\hat{\sigma}'(0)}\right) = 0$, we can generate a function obeying (a) - (c) by extending $\tilde{\varepsilon}(\lambda)$ to the complete domain $\lambda > 0$ via

$$\begin{aligned} \tilde{\varepsilon}(\lambda) &= 0, & \lambda &> \sqrt{\hat{\sigma}'(0)} , \\ \tilde{\varepsilon}(\lambda) &= \varepsilon_\infty, & 0 < \lambda &< \sqrt{\hat{\sigma}'(\varepsilon_\infty)} . \end{aligned} \tag{2.12}$$

It remains to determine whether or not (d) holds. Before doing so, however, we note that the <u>first</u> line segment of $\hat{\sigma}(\varepsilon)$ traverses an ε-interval including $[\beta_1, \beta_2]$ by virtue of $\sigma''(\varepsilon) < 0$ on $0 < \varepsilon < \beta_1$. Thus λ_1 is the phase boundary speed. The next N - 2 shock speeds λ_i are

$$\sqrt{\hat{\sigma}'\left(\tilde{\varepsilon}(\lambda_i -)\right)} = \lambda_i = \sqrt{\hat{\sigma}'\left(\tilde{\varepsilon}(\lambda_i +)\right)} , \quad (i = 2, \ldots, N - 1) , \tag{2.13}$$

since linear portions of $\hat{\sigma}(\varepsilon)$ strictly interior to $(0, \varepsilon_\infty)$ must join smoothly to points on $\sigma(\varepsilon)$. Thus the shocks $x = \lambda_i t$ $(i = 2, \ldots, N - 1)$ are transonic with respect to the material flanking it on both sides. If $N > 1$, the phase boundary speed λ_1 obeys

$$\sqrt{\sigma'\left(\tilde{\varepsilon}(\lambda_X -)\right)} = \lambda_1 > \sqrt{\sigma'\left(\tilde{\varepsilon}(\lambda_1 +)\right)} , \quad (N > 1) , \tag{2.14}$$

where the inequality is strict only if $\tilde{\varepsilon}(\lambda_1 +) = 0$. In such a case the phase boundary is supersonic with respect to the material in front. Similarly the fastest shock $x = \lambda_N t$ obeys

$$\sqrt{\sigma'(\tilde{\varepsilon}(\lambda_N-))} > \lambda_N = \sqrt{\sigma'(\tilde{\varepsilon}(\lambda_N+))}\ , \quad (N > 1)\ . \tag{2.15}$$

Finally, if the phase boundary is the only discontinuity front, then

$$\sqrt{\sigma'(\tilde{\varepsilon}(\lambda_1-))} > \lambda_1 > \sqrt{\sigma'(\varepsilon(\lambda_1+))}\ , \quad (N = 1)\ . \tag{2.16}$$

Turning now to the question of whether or not (d) holds, we note that nothing in the above construction prevents $\tilde{\varepsilon}(\lambda)$ from taking values in the range $[\varepsilon_J, \beta_1]$. We can, however, enforce (d), while maintaining (a) - (c), by repeating the same construction, not to the upper concave envelope of $\sigma(\varepsilon)$ on $0 \leq \varepsilon \leq \varepsilon_\infty$, but rather to the upper concave envelope on $0 \leq \varepsilon \leq \varepsilon_\infty$ of the "deleted" function

$$\sigma(\varepsilon_J;\varepsilon) = \begin{cases} \sigma(\varepsilon), & 0 < \varepsilon < \varepsilon_J,\ \bar{\varepsilon}_J < \varepsilon < \varepsilon_\infty, \\ \text{not defined*}, & \varepsilon_J < \varepsilon < \bar{\varepsilon}_J\ . \end{cases} \tag{2.17}$$

The program is now to define $\tilde{\varepsilon}(\lambda)$ in a fashion analagous to (2.11) by means of the equation

$$\lambda^2 = \hat{\sigma}_2\left(\varepsilon_J;\tilde{\varepsilon}(\lambda)\right)\ , \tag{2.18}$$

where $\hat{\sigma}_2 = \partial\hat{\sigma}(\varepsilon_J;\varepsilon)/\partial\varepsilon$. The only apparent difficulty is that, unlike $\hat{\sigma}(\varepsilon)$, $\hat{\sigma}(\varepsilon_J;\varepsilon)$ may be kinked at $\varepsilon = \varepsilon_J$, rendering $\sigma_2(\varepsilon_J;\varepsilon)$ discontinuous at this point. Examples of the two exhaustive alternatives:

$$\text{Alt. 1: } \hat{\sigma}(\varepsilon_J;\varepsilon_J) = \sigma_J\ , \qquad \text{Alt. 2: } \hat{\sigma}(\varepsilon_J;\varepsilon_J) > \sigma_J\ , \tag{2.19}$$

are shown in figure 3. No kink occurs for Alt. 2; indeed in this case $\hat{\sigma}(\varepsilon_J;\cdot) = \hat{\sigma}(\cdot)$ and the original construction (2.11) - (2.12) yields an $\tilde{\varepsilon}(\lambda)$ obeying (a) - (d). The

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*or, if one prefers, $\sigma(\varepsilon_J;\varepsilon) = -\infty$ on $\varepsilon_J < \varepsilon < \bar{\varepsilon}_J$

Then the solution $\phi \in S$ of (5.2) is continuously differentiable with $\mathring{\phi}(0) = 1$ and $0 < \mathring{\phi}(t) < 1$ for all $t > 0$.

The theorem will be proved in several stages. The hypotheses of Theorem 2 are assumed throughout.

<u>Lemma 1</u>: Let $A: B \to \mathbf{R}^+$, where the set $B \subset \mathbf{R}^+$ is such that $\phi(B) \subset B$ and $0 \in \bar{B}$. Let A obey

$$0 \leq A(t) \leq 1 \tag{5.16}$$

and

$$[A(\phi(t)) - 2 - \mathring{Q}(\phi(t))]A(t) + 1 = 0 \tag{5.17}$$

for all $t \in B$. If $t_k \in B$ and $t_k \to 0$, then $A(t_k) \to 1$.

<u>Proof</u>: If $A(t_k) \not\to 1$ we conclude from (5.16) the existence of a subsequence t_{k_ℓ} such that $A(t_{k_\ell}) \to \gamma_0$, $0 \leq \gamma_0 < 1$. Thus it is sufficient to show that if $A(t_k) \to \gamma_0$, then $\gamma_0 = 1$.

For each $n = 0,1,2,\ldots$, (5.3) yields

$$\lim_{k\to\infty} \phi^{[n]}(t_k) = 0 , \quad \lim_{k\to\infty} \mathring{Q}(\phi^{[n]}(t_k)) = 0 . \tag{5.18}$$

Since $\phi(B) \subset B$, (5.17) ensures the existence of each limit

$$\gamma_n = \lim_{k\to\infty} A(\phi^{[n]}(t_k)) , \quad n = 1,2,\ldots, \tag{5.19}$$

obeying

$$\gamma_n = 2 - 1/\gamma_{n-1} , \quad n = 1,2,\ldots, \tag{5.20}$$

which is the same as

$$1 - \gamma_n = (1 - \gamma_{n-1})/\gamma_{n-1} , \quad n = 1,2,\ldots \tag{5.21}$$

From (5.19), (5.16) and the definition of γ_0, we have $0 \leq \gamma_n \leq 1$ $(n = 0,1,\ldots)$, thus (5.21) gives $\gamma_n \leq \gamma_{n-1}$. Hence $\gamma_n \to \gamma$ with $1 - \gamma = (1 - \gamma)/\gamma$, so $\gamma = 1$. Collecting results we have $1 = \gamma = \lim_{n\to\infty} \gamma_n \leq \gamma_0 \leq 1$, whence $\gamma_0 = 1$. /

for all $n = 1,2,3...$ and all $t > 0$. Taking the limit of (5.11) as $n \to \infty$ and using (5.8) we arrive at

(5.12) $$|\phi(t) - \psi(t)| \leq |\phi(0) - \psi(0)| = 0 ,$$

hence $\psi = \phi$. /

Theorem 1 implies that $\mathring{\phi}$ exists a.e. on R^+ with $0 \leq \mathring{\phi} \leq 1$ and moreover that $\phi(t) = \int_0^t \mathring{\phi}(s)ds$, the integral being in the sense of Lebesgue. Our next aim is to show that ϕ is continuously differentiable and satisfies (4.21). Let $D \subset R^+$ be a Lebesgue measurable set with $\mu(D) = 0$ such that $\mathring{\phi}$ exists for all $t \in D^c$. Here and throughout the section, μ will denote Lebesgue measure and ^{c} will denote set complement in R^+. Differentiating (5.2) gives

(5.13) $$[\mathring{\phi}(\phi(t)) - 2 - \mathring{Q}(\phi(t))]\mathring{\phi}(t) + 1 = 0 ,$$

for all $t \in D^c$ such that $\phi(t) \in D^c$. Since $t \in D^c$ need not imply $\phi(t) \in D^c$, it is inconvenient to work with the sets D and D^c. Instead, we seek a set $F \subset R^+$ such that (i) $\mu(F) = 0$, (ii) $\mathring{\phi}$ exists for all $t \in F^c$ and (iii) $\phi(F^c) \subset F^c$. Clearly the set

(5.14) $$F = \bigcup_{n=0}^{\infty} \{(\phi^{-1})^{[n]}(D)\} ,$$

satisfies the latter two requirements. Thus (5.13) holds for all $t \in F^c$. To show that $\mu(F) = 0$ we note that $\phi \in S$ ensures that ϕ^{-1} is a strictly increasing function. Moreover, (5.2) gives explicitely

(5.15) $$\phi^{-1}(t) = 2t + Q(t) - \phi(t) .$$

Suppose Q is absolutely continuous. Then so is ϕ^{-1} which, in turn, renders $\mu(F) = 0$. In particular, Q is absolutely continuous under the hypotheses of

Theorem 2: Let Q be continuously differentiable with $Q(0) = \mathring{Q}(0) = 0$, $\mathring{Q}(z) > 0$ for $z > 0$ and, in addition, let $\mathring{Q}$ be Lipschitz on every interval $[0,z_0]$.

We are first going to show that the iterative scheme

$$\phi_n(t) = K^{-1}(L\phi_{n-1}(t)), \qquad \phi_1(t) = K^{-1}(\tfrac{1}{2}\, t)\ , \tag{5.7}$$

converges pointwise to a solution $\phi \in S$ of (5.5).

We have $0 \leq \phi_n(t) \leq t$ since $\phi_n \in S$. An inductive argument now gives $\phi_n(t) \geq \phi_{n-1}(t)$, thus $\phi_n(t) \to \phi(t)$ for all $t \geq 0$. Since $\phi_n \in S$, it follows that $\phi(t)$ is (i) Lipschitz with $\mathrm{Lip}[\phi] \leq 1$, (ii) increasing, and (iii) obeys $\phi(0) = 0$. Since $L\phi_n(t) \to L\phi(t)$ and $K\phi_n(t) \to K\phi(t)$ it is immediate from (5.7) that ϕ is a solution of (5.5) or equivalently (5.2). From (5.2) we have $t_1 \neq t_2$ implies $\phi(t_1) \neq \phi(t_2)$, whence ϕ is strictly increasing and so $\phi \in S$. In addition, for $t > 0$ we have $\phi(t) > 0$, so that $K^{-1}(L\phi(t)) < L\phi(t) \leq t$ by (5.4). Since $K^{-1}(L\phi(t)) = \phi(t)$, this establishes (5.3). It remains to show that ϕ is the unique solution of (5.2) in S.

It is immediate from (5.3) that

$$\lim_{n\to\infty} \phi^{[n]}(t) = 0\ , \qquad (t \geq 0)\ . \tag{5.8}$$

Suppose now that $\psi \in S$ is also a solution of (5.2).

Then

$$\begin{aligned} &|\phi(t) - \psi(t)| = |K^{-1}(L\phi(t)) - K^{-1}(L\psi(t))| \leq \\ &|L\phi(t) - L\psi(t)| = \tfrac{1}{2}\,|\phi(\phi(t)) - \psi(\psi(t))| \\ &\leq \tfrac{1}{2}\,|\phi(\phi(t)) - \psi(\phi(t))| + \tfrac{1}{2}\,|\psi(\phi(t)) - \psi(\psi(t))| \\ &\leq \tfrac{1}{2}\,|\phi(\phi(t) - \psi(\phi(t))| + \tfrac{1}{2}\,|\phi(t) - \psi(t)|\ , \end{aligned} \tag{5.9}$$

whence

$$|\phi(t) - \psi(t)| \leq |\phi(\phi(t)) - \psi(\phi(t))|\ . \tag{5.10}$$

This gives, by iteration,

$$|\phi(t) - \psi(t)| \leq |\phi(\phi^{[n]}(t)) - \psi(\phi^{[n]}(t))|\ , \tag{5.11}$$

5. Existence of a Solution to the Functional Equation

In this section, we prove that (4.14), (4.19) has a unique solution obeying (4.21). In addition, the first of the two theorems in this section furnishes an iterative scheme for solving (4.19).

Let $\mathbf{R}^+ = \{x | x \geq 0\}$ and for $n = 0,1,2, \ldots$ let $\phi^{[n]}(x) = \overbrace{\phi(\phi(\phi(\ldots(\phi(x))\ldots)))}^{n}$, $\phi^{[0]}(x) = x$. Define the set of functions

$$(5.1) \qquad \mathcal{S} = \{\phi : \mathbf{R}^+ \to \mathbf{R}^+ | \phi(0) = 0 , \quad \phi(t) \text{ strictly increasing and } \phi \text{ Lipschitz with } \mathrm{Lip}[\phi] \leq 1\} .$$

Note that $\mathcal{S}$ is closed under composition. The following theorem shows that (4.19) has a unique solution $\phi \in \mathcal{S}$ for a class of functions Q which include those Q obeying (4.14).

<u>Theorem 1</u>: Let $Q: \mathbf{R}^+ \to \mathbf{R}^+$ be continuous and strictly increasing with $Q(0) = 0$. Then the equation

$$(5.2) \qquad \phi(\phi(t)) - 2\phi(t) + t - Q(\phi(t)) = 0 , \quad (t \geq 0)$$

possesses a solution $\phi \in \mathcal{S}$. Moreover this solution ϕ obeys

$$(5.3) \qquad 0 < \phi(t) < t , \quad (t > 0)$$

and is the only solution of (5.2) in $\mathcal{S}$.

<u>Proof</u>: Define $L: \mathcal{S} \to \mathcal{S}$ by $L\psi(t) = \frac{1}{2}\psi(\psi(t)) + \frac{1}{2}t$. Also define $K: \mathbf{R}^+ \to \mathbf{R}^+$ by $K(z) = z + \frac{1}{2}Q(z)$. The hypotheses on Q ensure that K is invertible on $\mathbf{R}^+$ with the inverse function K^{-1} obeying

$$(5.4) \qquad 0 < K^{-1}(z) < z , \quad (z > 0) .$$

Furthermore, one may verify that $K^{-1} \in \mathcal{S}$.

We may write (5.2) either as

$$(5.5) \qquad K(\phi(t)) = L\phi(t) , \quad (t \geq 0) ,$$

or as

$$(5.6) \qquad \phi(t) = K^{-1}(L\phi(t)) , \quad (t \geq 0) .$$

Then (4.15) requires that

$$b(\phi(z)) = \frac{1}{2}\,\phi(z) + \frac{1}{2}\,z\ , \tag{4.17}$$

whereas (4.11) demands that

$$b(\phi(z)) = G(\phi(z)) - \frac{1}{2}\,\phi(\phi(z)) - \frac{1}{2}\,\phi(z), \qquad \left(b(\phi(z)) < \hat{t}\right)\ . \tag{4.18}$$

Consequently (4.12), (4.17-18) yield the equation

$$\phi(\phi(z)) - 2\phi(z) + z = Q(\phi(z))\ , \tag{4.19}$$

which is (1.5). In light of (4.16) and (4.9), the phase boundary trajectory $s(t)$ is given in terms of ϕ as

$$s(t) = c\left[a^{-1}(t) - t\right]\ , \quad a(t) = \frac{1}{2}\left[\phi(t) + t\right]\ , \quad t < \hat{t}\ , \tag{4.20}$$

where the restriction $t < \hat{t}$ follows from $b(\phi(z)) < \hat{t}$. Physically for $t > \hat{t}$, $\overset{+}{\varepsilon}(s(t),t) < \varepsilon_J$ and (4.8) no longer reduces to (4.10). Note that $s(t)$ completely determines the region U for $t < \hat{t}$ and, by virtue of (4.7), the displacement $u(x,t)$ in this region.

The derivation of (4.19) assumes that $s(t)$ is continuously differentiable and obeys (4.5). These conditions are equivalent to ϕ being continuously differentiable with

$$\phi(0) = 0, \quad \mathring{\phi}(0) = 1, \quad 0 < \mathring{\phi}(z) < 1 \quad \text{for} \quad z > 0\ . \tag{4.21}$$

Notice that (4.21) and the second of (4.20) yields $\mathring{a}(t) > 1/2$, which, in turn, guarantees the existence of $a^{-1}(t)$ which appears in the first of (4.20).

$h(t)$ generates acoustic waves which drive the phase boundary forward while ringing back and forth between it and the loading device. I am unaware of a general mathematical theory appropriate for treating this system in the above form. Fortunately, this problem simplifies considerably in the event that $\overset{+}{\varepsilon}(\cdot,\cdot)$ and $\overset{+}{v}(\cdot,\cdot)$ are individually constant. Recall that such a situation prevails immediately ahead of the phase boundary for all times $t < \hat{t}$. Consequently, for all times t such that $b(t) < \hat{t}$, one has $\overset{+}{\varepsilon}(s(b(t)),b(t)) = \overset{+}{\varepsilon}(s(a(t)),a(t)) = \varepsilon_J$, $\overset{+}{v}(s(b(t)),b(t)) = \overset{+}{v}(s(a(t)),a(t))$. This, in conjunction with $[\sigma(\varepsilon_J)-D]/c^2 = \bar{\varepsilon}_J$, reduces (4.8) to

$$h(t) = \frac{1}{2}\,[\bar{\varepsilon}_J - \varepsilon_J]\,[\dot{b}(t) - \dot{a}(t)] + \varepsilon_J, \qquad (b(t) < \hat{t}) \;. \tag{4.10}$$

This equation integrates immediately and, with the aid of $a(0) = b(0) = 0$, may be cast into the form

$$a(t) + b(t) = G(t), \qquad (b(t) < \hat{t}) \;, \tag{4.11}$$

where

$$G(t) = 2t + \frac{1}{2}\,Q(t), \quad Q(t) = [4/(\bar{\varepsilon}_J - \varepsilon_J)]\int_0^t [h(r) - \bar{\varepsilon}_J]dr \;. \tag{4.12}$$

We note for future reference that Q is of unlimited smoothness with

$$\dot{Q}(z) = 4[h(z) - \bar{\varepsilon}_J]/[\bar{\varepsilon}_J - \varepsilon_J] \;, \tag{4.13}$$

which in conjunction with (3.3) yields

$$Q(0) = \dot{Q}(0) = 0, \quad \dot{Q}(z) > 0 \quad (z > 0) \;. \tag{4.14}$$

Return now to (4.9) and note that (4.5) ensures that $a(t)$ and $b(t)$ are each continuously differentiable and monotonically increasing. Let $a^{-1}(\cdot)$, $b^{-1}(\cdot)$ denote their respective inverse functions. Eliminating $s(\cdot)$ between (4.9) furnishes

$$a^{-1}(t) + b^{-1}(t) = 2t \;. \tag{4.15}$$

We introduce

$$\phi(z) = 2a(z) - z \;, \qquad z > 0 \;. \tag{4.16}$$

One may express $u(s(t_a),t_a)$ and $u(s(t_b),t_b)$* as line integrals of $\varepsilon(s(\eta)-,\eta)$, $v(s(\eta)-,\eta)$ along $\xi = s(\eta)$. Upon subsequently eliminating $\varepsilon(s(\eta)-,\eta)$, $v(s(\eta)-,\eta)$ in favor of $\overset{+}{\varepsilon}(s(\eta),\eta)$, $\overset{+}{v}(s(\eta),\eta)$ by means of (1.4) and (4.1) one arrives at

$$
\begin{aligned}
u(x,t) = u(0,0) &+ \int_0^{t_a} \{\mathring{s}(\eta)\overset{+}{\varepsilon}(s(\eta),\eta) + \overset{+}{v}(s(\eta),\eta)\}d\eta \\
&- \frac{1}{2c}\int_{t_a}^{t_b} \{[\mathring{s}(\eta)-c]\overset{+}{v}(s(\eta),\eta) - c\mathring{s}(\eta)\overset{+}{\varepsilon}(s(\eta),\eta) + \sigma(\overset{+}{\varepsilon}(s(\eta),\eta))-D\}d\eta
\end{aligned} \tag{4.7}
$$

The function $s(\eta)$ appearing in (4.7) is as yet unknown; it must be chosen so as to satisfy the remaining condition $\varepsilon(0,t) = h(t)$. With a view toward expressing $\varepsilon(0,t)$ in terms of quantities along $\xi = s(\eta)$, we differentiate (4.7) with respect to x and subsequently set $x = 0$. This exercise essentially reduces to a calculation of $\frac{\partial t_a}{\partial x}\big|_{x=0}$ and $\frac{\partial t_b}{\partial x}\big|_{x=0}$. These may be expressed in terms of $\mathring{a}(t)$ and $\mathring{b}(t)$, where $a(t) \equiv t_a(0,t)$ and $b(t) \equiv t_b(0,t)$. The result of these calculations is

$$
\begin{aligned}
h(t) = &\frac{1}{2}\{[\sigma(\overset{+}{\varepsilon}(s(b(t)),b(t)))-D]/c^2 - \overset{+}{\varepsilon}(s(b(t)),b(t))\}\mathring{b}(t) \\
&+ \frac{1}{2}\{[\sigma(\overset{+}{\varepsilon}(s(a(t)),a(t)))-D]/c^2 - \overset{+}{\varepsilon}(s(a(t)),a(t))\}\mathring{a}(t) \\
&+ \frac{1}{2}\{\overset{+}{\varepsilon}(s(a(t)),a(t)) + \overset{+}{\varepsilon}(s(b(t)),b(t)) + [\overset{+}{v}(s(a(t)),a(t)) \\
&\qquad - \overset{+}{v}(s(b(t)),b(t))]/c^2\} .
\end{aligned} \tag{4.8}
$$

The functions $a(t)$ and $b(t)$ are connected to $s(t)$ by means of the implicit relations

$$
s(a(t)) - c[t - a(t)] = 0, \quad s(b(t)) + c[t - b(t)] = 0, \quad a(0) = b(0) = 0 . \tag{4.9}
$$

Equations (4.8), (4.9) for the unknowns $s(\cdot)$, $a(\cdot)$, $b(\cdot)$ describe how the end-strain

---

*which are continuous across $\xi = s(\eta)$ by virtue of the first of (1.4).

values behind the phase boundary by means of the discontinuity conditions (1.4). With the aid of (4.1), these discontinuity relations can be solved for $\varepsilon(s(\eta)-,\eta)$ and $v(s(\eta)-,\eta)$ provided that $\overset{\circ}{s}(\eta) \neq c$. In the interior of U, the dynamical fields obey (1.1), which becomes

$$\frac{\partial^2 u}{\partial \eta^2} = c^2 \frac{\partial^2 u}{\partial \xi^2}, \tag{4.4}$$

by virtue of (4.1).

We assume, subject to later verification, that $s(\eta)$ is continuously differentiable with

$$s(0) = \overset{\circ}{s}(0) = 0 \quad \text{and} \quad 0 < \overset{\circ}{s}(\eta) < c \quad \text{for} \quad \eta > 0 . \tag{4.5}$$

The standard representation for solutions of (4.4) in terms of Cauchy data on a given curve furnishes

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\{u(s(t_a),t_a) + u(s(t_b),t_b)\} \\ &\quad - \frac{1}{2c}\int_{t_a}^{t_b} \{v(s(\eta)-,\eta)\overset{\circ}{s}(\eta) + c^2\varepsilon(s(\eta)-,\eta)\}d\eta \end{aligned} \tag{4.6}$$

for all $(x,t) \in U$. Here, $t_a = t_a(x,t)$ and $t_b = t_b(x,t)$ delimit the domain of dependence upon $\xi = s(\eta)$ as depicted in figure 4.

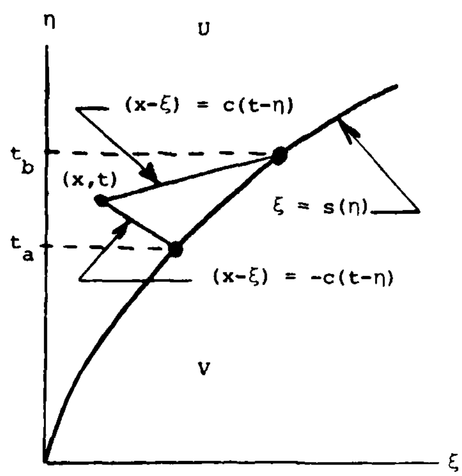

Figure 4. Graphical representation of the times $t_b = t_b(x,t)$ and $t_a = t_a(x,t)$ for $(x,t) \in U$. These times are also given implicitely as the roots of $x = s(t_b) + c[t - t_b]$ and $x = s(t_a) - c[t - t_a]$ respectively.

4. A Functional Equation

For the remainder of this paper we restrict attention to materials for which

$$\text{(4.1)}\qquad \begin{aligned} &\text{i)}\quad \sigma_J < \sigma_\beta, \\ &\text{ii)}\quad \sigma(\varepsilon) = c^2\varepsilon + D\ , \qquad \varepsilon > \tilde{\varepsilon}_J\ . \end{aligned}$$

Such "second phase linear" materials have been studied previously in [11] in connection with this problem in which $\sigma_J = \sigma_\beta$. We shall show that there exists a time $\hat{t} > t^*$, and possibly infinite, such that $P(\hat{t})$ has a solution with an energy confining phase boundary, the location of which is governed by (1.5).

By transforming the time variable $t \to t - t^*$ we may, without loss of generality, set $t^* = 0$. Let (ξ,η) be a point in the (x,t)-plane and let V and U be respectively the regions in front and behind the phase boundary. In terms of (ξ,η), we have

$$\text{(4.2)}\qquad \begin{aligned} V &= \{(\xi,\eta)\ |\ \xi > s(\eta),\ \ \eta > 0\}\ , \\ U &= \{(\xi,\eta)\ |\ 0 < \xi < s(\eta),\ \ \eta > 0\}\ . \end{aligned}$$

The fundamental difference between U and V is that whereas the dynamical fields ε and v in U remain to be found, these fields in V are determined by events occurring at $x = 0$ before the time of phase boundary emergence. To acknowledge this distinction, we shall let $\overset{+}{\varepsilon}$ and $\overset{+}{v}$ be respectively the (known) strain and velocity in the region V. In particular, $\overset{+}{\varepsilon}$ and $\overset{+}{v}$ are individually constant in that portion of V between the phase boundary $\xi = s(\eta)$ and the characteristic curve $\xi = \sqrt{\sigma'(\varepsilon_J)}\,\eta$. Should the phase boundary overtake the front of this region, we shall denote by $\hat{t}$ the time at which this event occurs. Thus $\hat{t} > 0$ is the (first positive) root — if it exists — of

$$\text{(4.3)}\qquad s(\hat{t}) = \sqrt{\sigma'(\varepsilon_J)}\ \hat{t}\ , \qquad \hat{t} > 0\ .$$

We shall say that $\hat{t} = \infty$ in the event that the phase boundary never overtakes this front.

For all values $\eta > 0$ the strain and velocity immediately in front of the phase boundary — $\varepsilon(s(\eta)+,\eta) = \overset{+}{\varepsilon}(s(\eta),\eta)$ and $v(s(\eta)+,\eta) = \overset{+}{v}(s(\eta),\eta)$ — are functions of the values η and $s(\eta)$. This strain and velocity is in turn related to the corresponding

particular, the characteristic C^+ which would issue from $(x,t) = (0,t^*)$ is the ray $x = \sqrt{\sigma'(\varepsilon_J)}\,[t - t^*]$.

At the time $t = t^*$ a phase boundary -- $x = s(t)$, say -- emerges at $x = 0$ and subsequently propagates into the interior. By virue of (1.4), the phase boundary velocity obeys

$$\dot{s}(t) = \left\{\frac{\sigma(\varepsilon(s(t)+,t)) - \sigma(\varepsilon(s(t)-,t))}{\varepsilon(s(t)+,t) - \varepsilon(s(t)-,t)}\right\}^{1/2}. \tag{3.4}$$

This relation, in conjunction with the continuity of $g(t)$ at $t = t^*$, yields $\dot{s}(t^*) = 0$. Thus there exists a region in the (x,t)- plane

$$s(t) < x < \sqrt{\sigma'(\varepsilon_J)}\,[t - t^*] , \tag{3.5}$$

immediately in front of the phase boundary, which is conspicuous because the solution within this region is not as yet determined. We now inquire as to the value of the strain in this region.

A first alternative is that the value of $\varepsilon(x,t)$ in this region could exceed the value ε_J and yet remain below the value β_1. If such is the case, then events at the external boundary occurring after $t = t^*$ should play some role in determining $\varepsilon(x,t)$ in this region. A second alternative -- that which shall be pursued here -- is that the strain in this region remains fixed at the value ε_J. In this case, the velocity in this region must also be constant, its value being given by the (constant) velocity on the boundary characteristic $x = \sqrt{\sigma'(\varepsilon_J)}\,[t - t^*]$. Unlike the first alternative, this situation is one in which events at the external boundary occurring before $t = t^*$ completely determine the fields in front of the phase boundary. What is not determined is the location of this boundary itself, as well as the fields behind it. Physically this corresponds to a situation in which all of the energy delivered to the bar after the emergence of the phase boundary remains confined behind the phase boundary. We shall use the expression <u>energy confining phase boundary</u> for this second state of affairs. In the next section we formulate a free boundary problem for such a phase boundary and show that it leads to equation (1.5).

3. Energy Confining Phase Boundaries in the Presence of Smooth Loads

We now focus attention upon loads $g(t)$ which are of unlimited smoothness with $g(0) = 0$, $\dot{g}(t) > 0$ for $0 \leq t \leq t^*$ and $g(t) > \sigma_J$ for $t > t^*$. It will be convenient to write the boundary condition (1.2) as

$$\varepsilon(0,t) = h(t) \; ; \tag{3.1}$$

where $h(t)$ is the end-strain, obeying (2.1), associated with the traction $g(t)$. It is given uniquely as the solution to

$$g(t) = \sigma(\varepsilon_J;h(t)) \; . \tag{3.2}$$

The function $h(t)$ of (3.2) is discontinuous at $t = t^*$, while at all other values t it is of unlimited smoothness. Specifically it obeys

$$\begin{aligned} &h(0) = 0, \quad \dot{h}(t) > 0 \qquad (0 \leq t \leq t^*), \\ &h(t^*-) = \varepsilon_J, \qquad h(t^*+) = \bar{\varepsilon}_J, \qquad h(t) > \varepsilon_J \quad (t > t^*) \; . \end{aligned} \tag{3.3}$$

In terms of $h(t)$, the problem we seek to solve is given up to any time $T > 0$ as

$$P(T) \begin{cases} \text{Find } u\text{: } (x > 0,\ 0 \leq t < T) \to R. \text{ which is twice} \\ \text{continuously differentiable except across curves where} \\ \text{possible discontinuities in } \varepsilon = \partial u/\partial x \text{ and } v = \partial u/\partial t \\ \text{are to be restricted by (1.4). Otherwise } u,\ \varepsilon \text{ and } v \\ \text{are to obey (1.1) on } (x > 0,\ 0 < t < T),\ (1.3) \text{ on } x > 0,\ t = 0 \\ \text{and (3.1) on } x = 0;\ 0 \leq t < T. \end{cases}$$

We are interested in solving $P(T)$ for $T > t^*$. Notice that the problem $P(t^*)$ describes the situation before and up to the emergence of the phase boundary; it is easily solved by the method of characteristics. The following two sentences summarize the pertinent features of this solution; a complete development may be found in [11]. In brief the solution for $T \leq t^*$ relies upon the condition $\sigma''(\varepsilon) < 0$ for $\varepsilon < \varepsilon_J$ in conjunction with $\dot{h}(t) > 0$ to ensure that the characteristics C^+ — curves obeying $\frac{dx}{dt} = \sqrt{\sigma'(\varepsilon(x,t))}$ — do not intersect; the resulting smooth solution is one in which each member of C^+ is a straight ray upon which the strain and velocity are constant. In

say $\bar{\bar{\varepsilon}}(\varepsilon^*,\lambda)$ where $\bar{\bar{\varepsilon}}(\tilde{\varepsilon}(\lambda_1+),\lambda) = \tilde{\varepsilon}(\lambda)$. All the solutions for $\varepsilon^* < \tilde{\varepsilon}(\lambda_1+)$ have phase boundaries which travel faster than that of the solution $\tilde{\varepsilon}(\lambda)$.

Now suppose that $\tilde{\varepsilon}(\lambda_1-) < \varepsilon_\infty$. Then the upper concave envelope of $\sigma(\varepsilon)$ deleted on $(\varepsilon_J,\varepsilon^{**})$ with $\tilde{\varepsilon}(\lambda_1-) < \varepsilon^{**} < \varepsilon_\infty$ again differs from $\hat{\sigma}(\varepsilon_J;\varepsilon)$ and the resulting solutions all have slower moving phase boundaries than that of the solution $\tilde{\varepsilon}(\lambda)$. Solutions with more than N discontinuities can result from deleting $\sigma(\varepsilon)$ on additional intervals. Finally, the extreme case of deleting everything but the end-points $\varepsilon = 0$ and $\varepsilon = \varepsilon_\infty$ yields an upper concave envelope which is the line segment connecting $(0,0)$ to $(\varepsilon_\infty,\sigma_\infty)$. The associated solution is

$$\bar{\bar{\varepsilon}}(\lambda) = \begin{cases} 0\,, & \lambda > \sqrt{\sigma_\infty/\varepsilon_\infty}\,, \\ \varepsilon_\infty\,, & 0 < \lambda < \sqrt{\sigma_\infty/\varepsilon_\infty}\,, \end{cases}$$

$$\bar{\bar{v}}(\lambda) = \begin{cases} 0\,, & \lambda > \sqrt{\sigma_\infty/\varepsilon_\infty}\,, \\ -\sqrt{\sigma_\infty\varepsilon_\infty}\,, & 0 < \lambda < \sqrt{\sigma_\infty/\varepsilon_\infty}\,. \end{cases} \tag{2.26}$$

As is the case with $\tilde{\varepsilon}$, these other solutions $\bar{\bar{\varepsilon}}$ do not satisfy all of the inequalities (2.13) - (2.16) unless $\hat{f}(\varepsilon) = \hat{\sigma}(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_\infty$.

In summary, whenever $\hat{\sigma}(\varepsilon_J;\varepsilon)$ differs from the line segment connecting (0,0) to $(\varepsilon_\infty,\sigma_\infty)$, there exist solutions other than $\tilde{\varepsilon}(\lambda)$, $\tilde{v}(\lambda)$ for the impulsive load (2.3). In particular, a family of solutions occurs for Alt. 1, since then $\tilde{\varepsilon}(\lambda_1+) = \varepsilon_J > 0$. A pertinent question is: which, if any, of these solutions, gives an asymptotic description of the dynamical fields in a bar subject to a smooth load $g(t)$ with $g(t) \to \sigma_\infty$? In what follows it will be shown under certain conditions for a special class of materials that the problem with a smooth load is reducible to equation (1.5). One result of examining this equation asymptotically will be that if $g(t) \to \sigma_\infty$ and Alt. 1 holds, then $\tilde{\varepsilon}(x/t)$ is the x/t fixed, $t \to \infty$ limit of the resulting strain field.

Finally, $\tilde{\varepsilon}(\lambda)$ obeys (2.5) whenever the graph of $\hat{\sigma}_2(\varepsilon_J;\varepsilon)$ is neither horizontal nor vertical. Thus extending $\tilde{\varepsilon}(\lambda)$ to $\lambda > 0$ by

$$\tilde{\varepsilon}(\lambda) = 0\,, \qquad \lambda > \sqrt{\hat{\sigma}_2(\varepsilon_J;0)}\,,$$

(2.22)

$$\tilde{\varepsilon}(\lambda) = \varepsilon_\infty\,, \qquad 0 < \lambda < \sqrt{\hat{\sigma}_2(\varepsilon_J;\varepsilon_\infty)}\,,$$

yields an $\tilde{\varepsilon}(\lambda)$ which fulfills all the requirements (a) - (d).

An interesting feature of this solution is that the phase boundary may travel slower than the sound speed of the material into which it is proceeding. Although (2.13), (2.15) remain in force for ordinary shocks, (2.14), (2.16) continue to hold only for Alt. 2. For Alt. 1, (2.14) is replaced by

$$\text{(2.23)} \qquad \sqrt{\sigma'(\tilde{\varepsilon}(\lambda_1-))} = \lambda_1 < \sqrt{\sigma'(\tilde{\varepsilon}(\lambda_1+))} = \sqrt{\sigma'(\varepsilon_J)}\,, \quad (N > 1,\ \text{Alt. } 1)\,,$$

whereas (2.16) is changed to

$$\text{(2.24)} \qquad \lambda_1 < \sqrt{\sigma'(\tilde{\varepsilon}(\lambda_1-))}\,, \quad \lambda_1 < \sqrt{\sigma'(\tilde{\varepsilon}(\lambda_1+))} = \sqrt{\sigma'(\varepsilon_J)}\,, \quad (N = 1,\ \text{Alt. } 1)\,.$$

We now reserve the notation $\tilde{\varepsilon}(\lambda)$ for that particular solution of (a) - (d) given by (2.17), (2.18), (2.22) and enquire into the uniqueness of this solution.

The construction leading to $\tilde{\varepsilon}(\lambda)$ succeeded in satisfying (d) while maintaining (a) - (c) by "deleting" $\sigma(\varepsilon)$ on $\varepsilon_J < \varepsilon < \bar{\varepsilon}_J$ before taking the upper concave envelope. Indeed, if even more of $\sigma(\varepsilon)$ is deleted, giving say $f(\varepsilon)$, then the ensuing concave envelope $\hat{f}(\varepsilon)$ leads to a solution $\bar{\bar{\varepsilon}}(\lambda)$ of (a) - (d) by the same procedure applied to

$$\text{(2.25)} \qquad \lambda^2 = \hat{f}(\bar{\bar{\varepsilon}}(\lambda))\,.$$

Thus, for example, if $\tilde{\varepsilon}(\lambda_1+) > 0$, then the upper concave envelope of $\sigma(\varepsilon)$ deleted on $(\varepsilon^*,\bar{\varepsilon}_J)$, with $0 < \varepsilon^* < \tilde{\varepsilon}(\lambda_1+)$, differs from $\hat{\sigma}(\varepsilon_J;\varepsilon)$ and the resulting $\bar{\bar{\varepsilon}}(\lambda)$ differs from $\tilde{\varepsilon}(\lambda)$. Moreover each choice of ε^* on the interval $0 < \varepsilon^* < \tilde{\varepsilon}(\lambda_1+)$ generates a different upper concave envelope $\hat{f}(\lambda)$ so that ε^* parameterizes a family of solutions,

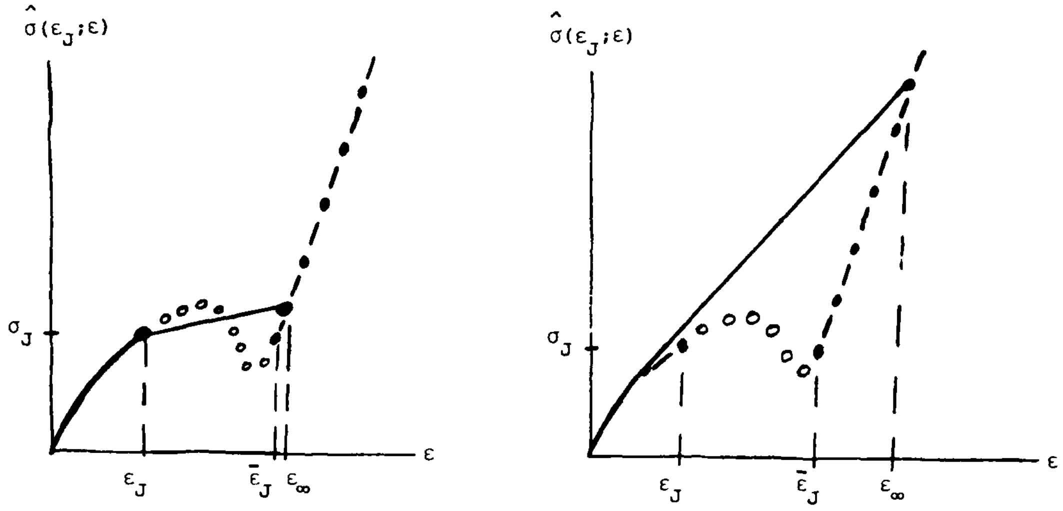

Figure 3. The upper concave envelope $\hat{\sigma}(\varepsilon_J;\varepsilon)$ *for two different values of* ε_∞. This envelope need only be above the dashed curve, which is $\sigma(\varepsilon_J;\varepsilon)$. The left and right figures are examples of Alt. 1 and Alt. 2 respectively.

program, however must be modified for Alt. 1, because it is generic for this alternative that

$$\sigma'(\varepsilon_J) = \hat{\sigma}_2(\varepsilon_J;\varepsilon_J-) > \hat{\sigma}_2(\varepsilon_J,\varepsilon_J+) \ . \tag{2.20}$$

In the face of (2.20) let us agree to identify $\hat{\sigma}_2(\varepsilon_J;\varepsilon_J)$ with the full interval $[\hat{\sigma}_2(\varepsilon_J;\varepsilon_J+),\ \hat{\sigma}_2(\varepsilon_J;\varepsilon_J-)]$. The graph of $\hat{\sigma}_2(\varepsilon_J;\varepsilon)$ will then be continuously decreasing on $0 < \varepsilon < \varepsilon_\infty$ and will involve a vertical segment at $\varepsilon = \varepsilon_J$. Hence (2.18) defines a piecewise continuous $\tilde{\varepsilon}(\lambda)$ *on* $\sqrt{\hat{\sigma}_2(\varepsilon_J;\varepsilon_\infty)} < \lambda < \sqrt{\hat{\sigma}_2(\varepsilon_J;0)}$. If $\lambda_1 > \lambda_2 > \dots > \lambda_N$ are the values where $\tilde{\varepsilon}(\cdot)$ is discontinuous, then λ_i^2 is the height of the $i^{\underline{th}}$ horizontal portion of $\hat{\sigma}_2(\varepsilon_J;\varepsilon)$. On the other hand, the vertical segment of $\hat{\sigma}_2(\varepsilon_J;\varepsilon)$ at $\varepsilon = \varepsilon_J$ yields

$$\tilde{\varepsilon}(\lambda) = \varepsilon_J\ , \quad \sqrt{\hat{\sigma}_2(\varepsilon_J;\varepsilon_J+)} < \lambda < \sqrt{\sigma'(\varepsilon_J)} \ . \tag{2.21}$$

Two sets which satisfy the requirements upon B stated above are F^c and R^+. Note that $\overset{\circ}{\phi}$ obeys (5.16), (5.17) for $B = F^c$. Thus we draw three corollaries central to the following development

Corollary 1: All A: $F^c \to R^+$ obeying (5.16), (5.17) on F^c have the property that $A(t_k) \to 1$ whenever $t_k \to 0$, $t_k \in F^c$.

Corollary 2: $\overset{\circ}{\phi}(t_k) \to 1$ for all $t_k \to 0$, $t_k \in F^c$.

Corollary 3: If A: $R^+ \to R^+$ obeys (5.16), (5.17) on R^+, then A is continuous at the origin with $A(0) = 1$.

The first two corollaries guarantee that $\overset{\circ}{\phi}$ is the unique solution of (5.17) in the following sense

Lemma 2: If A: $R^+ \to R^+$ obeys (5.16), (5.17) for all $t \in F^c$, then

$$A(t) = \overset{\circ}{\phi}(t) \quad \text{for all } t \in F^c.$$

Proof: Write (5.17) in the form $A(t) = -1/[A(\phi(t)) - 2 - \overset{\circ}{Q}(\phi(t))]$; a corresponding result holds with $\overset{\circ}{\phi}$ replacing A. By subtracting these two expressions and invoking (5.16), $0 < \overset{\circ}{\phi} < 1$, and $\overset{\circ}{Q}(\phi(t)) > 0$, we arrive at

$$(5.22) \qquad |A(t) - \overset{\circ}{\phi}(t)| < |A(\phi(t)) - \overset{\circ}{\phi}(\phi(t))| , \quad (t \in F^c) ,$$

which gives, by induction since $\phi(F^c) \subset F^c$,

$$(5.23) \qquad |A(t) - \overset{\circ}{\phi}(t)| < |A(\phi^{[n]}(t)) - \overset{\circ}{\phi}(\phi^{[n]}(t))| , \quad (t \in F^c,\ n = 1,2,\ldots) .$$

Letting $n \to \infty$, we have from (5.8) and Corollaries 1 and 2,

$$(5.24) \qquad |A(t) - \overset{\circ}{\phi}(t)| < \lim_{\substack{r \to 0 \\ r \in F^c}} |A(r) - \overset{\circ}{\phi}(r)| = 0 , \quad (t \in F^c) . /$$

We already have that $\overset{\circ}{\phi}$ is a solution of (5.17) on F^c. We now show that (5.17) possesses a solution not only on F^c, but also on R^+.

Lemma 3: There exists A: $R^+ \to R^+$, continuous with $A(0) = 1$, $0 < A(t) < 1$ for $t > 0$ such that (5.17) holds for all $t \in R^+$.

<u>Proof</u>: Consider the sequence of functions generated by

$$(5.25)\qquad A_{k+1}(t) = 1/\{2 + \mathring{Q}(\phi(t)) - A_k(\phi(t))\}\ , \qquad A_1(t) = 1/\{2 + \mathring{Q}(\phi(t))\}\ .$$

Separate arguments based upon induction and $\mathring{Q}(z) \geq 0$ yield $0 < A_k(t) \leq 1$, $(t \geq 0)$, and $A_{k+1}(t) \geq A_k(t)$, $(t \geq 0)$. Hence $A(t) \equiv \lim_{k\to\infty} A_k(t)$ exists and obeys $0 \leq A(t) \leq 1$. It is evident from (5.25) that $A(t)$ is a solution of (5.17). Also $A(0) = 1/\{2 + \mathring{Q}(\phi(0)) - A(\phi(0))\} = 1/\{2 - A(0)\}$ so that $A(0) = 1$. Moreover, for $t > 0$, the strict inequality $\mathring{Q}(\phi(t)) > 0$ furnishes $1/A(t) = 2 + \mathring{Q}(\phi(t)) - A(\phi(t)) > 2 - A(\phi(t)) \geq 1$, whence $0 < A(t) < 1$ for $t > 0$. The continuity of A at the origin is the result of Corollary 3. It remains to show that $A(t)$ is continuous for $t > 0$.

Let $t > 0$ and $\varepsilon > 0$ be given; we shall show that there exists $\delta > 0$ such that if $\tau > 0$ with $|t - \tau| < \delta$ then $|A(t) - A(\tau)| < \varepsilon$. Let q be a Lipschitz constant for $\mathring{Q}$ on $0 \leq z \leq t + 1$. Choose $d > 0$ such that $0 \leq x \leq d$, $0 \leq y \leq d$ implies $|A(x) - A(y)| < \varepsilon/2$. Now (5.8) guarantees the existence of an N such that $\phi^{[N]}(t + 1) < d$. From (5.17) and the bounds $A(z) \leq 1$, $\mathring{Q}(z) \geq 0$, it follows that

$$(5.26)\qquad |A(t) - A(\tau)| \leq |\mathring{Q}(\phi(\tau)) - \mathring{Q}(\phi(t))| + |A(\phi(t)) - A(\phi(\tau))|, \qquad (t \geq 0,\ \tau \geq 0)\ .$$

Iterating this inequality $N - 1$ times, we arrive at

$$(5.27)\qquad |A(t) - A(\tau)| \leq \Big\{ \sum_{k=1}^{N} |\mathring{Q}(\phi^{[k]}(t)) - \mathring{Q}(\phi^{[k]}(\tau))| \Big\} + |A(\phi^{[N]}(t)) - A(\phi^{[N]}(\tau))|\ .$$

Let $\delta = \min\{1, \varepsilon/2Nq\}$ and suppose that $|t - \tau| < \delta$. Then (5.27) implies

$$(5.28)\qquad |A(t) - A(\tau)| \leq \Big\{ \sum_{k=1}^{N} q|t - \tau| \Big\} + \frac{\varepsilon}{2} < \varepsilon\ .$$

We now rapidly complete the

<u>Proof of Theorem 2</u>: Let A be as given in Lemma 3. Then Lemmas 2 and 3 furnish, since $\mu(F) = 0$,

$$\phi(t) = \int_0^t \mathring{\phi}(s)ds = \int_0^t A(s)ds \ . \tag{5.29}$$

In view of the continuity of A, the latter integral is a Riemann integral. Hence, by the Fundamental Theorem of Integral Calculus, $\mathring{\phi}(t)$ exists for all $t > 0$ and $\mathring{\phi}(t) = A(t)$. Finally, Lemma 3 ensures all the requisite properties of $\mathring{\phi}$./

6. Large Time Behavior of Solutions

Theorems 1 and 2 guarantee the existence of a solution $\phi(t)$ to the system (4.14), (4.19), (4.21). The function ϕ, in turn, enables one to determine the region $\mathcal{U}$ and the function $u(x,t)$ in $\mathcal{U}$ by means of (4.6), (4.20). Thus we arrive at a solution involving an energy confining phase boundary to $T(\hat{t})$ for the class of materials (4.1). Here $\hat{t}$ is found from (4.3). Thus the question arises, what circumstances — if any — give rise to a problem for which $\hat{t} = \infty$? Or, equivalently, in terms of the known fields in front of the phase boundary: what conditons give rise to a phase boundary which never overtakes the front of the constant-strain region?

Since the leading edge of this front travels at speed $\sqrt{\sigma'(\varepsilon_J)}$, a partial answer to the above question is immediate from the bound $\mathring{s}(t) < c$. Namely, if $\sqrt{\sigma'(\varepsilon_J)} > c$, then $\mathring{s}(t) < \sqrt{\sigma'(\varepsilon_J)}$ which yields $\hat{t} = \infty$. On the other hand, for materials with $\sqrt{\sigma'(\varepsilon_J)} < c$, we will now show that $\hat{t} < \infty$ provided $h(t)$ becomes sufficiently large.

Consider $h(t) \to \varepsilon_\infty > \bar{\varepsilon}_J$ as $t \to \infty$. Then from (4.12) it follows that

$$\mathcal{Q}(\phi(t)) \to 4\left(\frac{\varepsilon_\infty - \bar{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}\right)\phi(t)\ , \tag{6.1}$$

whence (4.19) yields, to leading order as $t \to \infty$, the asymptotic relation

$$\phi(\phi(t)) + t \sim \left\{2 + 4\left(\frac{\varepsilon_\infty - \bar{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}\right)\right\}\phi(t) = [\gamma + 2]\phi(t)\ , \tag{6.2}$$

where we have introduced $\gamma = 4[\varepsilon_\infty - \bar{\varepsilon}_J]/[\varepsilon_\infty - \varepsilon_J] > 0$. The equation $\phi(\phi(t)) - [\gamma + 2]\phi(t) + t = 0$ has two solutions, $\phi(t) = \frac{1}{2}[\gamma + 2 \pm \sqrt{\gamma^2 + 4\gamma}]t$. However, since $\gamma > 0$, only the negative root yields a $\phi(t)$ obeying $0 < \phi(t) < t$ for $t > 0$. Hence

$$\phi(t) \sim \frac{1}{2}[\gamma + 2 - \sqrt{\gamma^2 + 4\gamma}]\ . \tag{6.3}$$

Assume now, for the sake of argument, that $\hat{t} = \infty$. Then (4.20) leads to

$$s(t) \sim \left(\frac{-\gamma + \sqrt{\gamma^2 + 4\gamma}}{\gamma + 4 - \sqrt{\gamma^2 + 4\gamma}}\right) t = \sqrt{\frac{\varepsilon_\infty - \bar{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}}\ ct\ ,$$

(6.4)

$$\mathring{s}(t) \sim \sqrt{\frac{\varepsilon_\infty - \bar{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}}\ c\ .$$

In view of (4.3), (6.4) a necessary condition for the assumption $\hat{t} = \infty$ to hold is

$$\sqrt{\frac{\varepsilon_\infty - \bar{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}}\ c < \sqrt{\sigma'(\varepsilon_J)}\ . \tag{6.5}$$

This condition is satisfied

$$\text{for}\quad \bar{\varepsilon}_J < \varepsilon_\infty < \varepsilon_T \quad \text{if}\quad \sqrt{\sigma'(\varepsilon_J)} < c\ ,$$

(6.6)

$$\text{for all}\quad \varepsilon_\infty > \bar{\varepsilon}_J \quad \text{if}\quad \sqrt{\sigma'(\varepsilon_J)} > c\ ,$$

where the value

$$\varepsilon_T \equiv \frac{\bar{\varepsilon}_J c^2 - \varepsilon_J \sigma'(\varepsilon_J)}{c^2 - \sigma'(\varepsilon_J)}\ . \tag{6.7}$$

The second of (6.6) was anticipated from the previous discussion. The first of (6.6) provides a boundary data upper threshold which is necessary for the phase boundary to remain behind the front of the constant strain region; hence it is a necessary condition for $\hat{t} = \infty$. We do not enquire in detail as to conditions sufficient to ensure that $\hat{t} = \infty$ for the case $\sqrt{\sigma'(\varepsilon_J)} < c$, other than to remark that it is evident from (4.3) that if $\mathring{s}(t) < \sqrt{\sigma'(\varepsilon_J)}$ for all $t > 0$, then $\hat{t} = \infty$. Thus $\varepsilon_\infty < \varepsilon_T$ is also a sufficient condition for $\hat{t} = \infty$ whenever $\ddot{s}(t) > 0$ for all $t > 0$. We conjecture that a condition sufficient to preclude a decellerating phase boundary is $\mathring{h}(t) > 0$ for all $t > 0$.

For the cases $h(t) \to \varepsilon_\infty > \overline{\varepsilon}_J$ with $\hat{t} = \infty$, we may calculate the large time strain and velocity fields for the problem $P(\infty)$, (4.1), in the limit $t \to \infty$ with $\lambda = x/t$ fixed. As mentioned in Section 3, these fields in the region $x > \sqrt{\sigma'(\varepsilon_J)}\,[t - t^*]$, are calculated in reference [11]. Whence we verify for $x > \{[\varepsilon_\infty - \varepsilon_J]/[\overline{\varepsilon}_\infty - \varepsilon_J]\}^{1/2} ct \sim s(t)$, that ε and v admit the expansions $\varepsilon(x,t) = \varepsilon(\lambda t,t) = \varepsilon_1(\lambda) + \varepsilon_2(\lambda,t)$, $v(x,t) = v_1(\lambda) + v_2(\lambda,t)$ where $\varepsilon_2, v_2 \to 0$ as $t \to \infty$ for fixed λ, and

(i) $\varepsilon_1(\lambda) = 0, \quad v_1(\lambda) = 0, \qquad \sqrt{\sigma'(0)} < \lambda < \infty$,

(ii) $\varepsilon_1(\lambda) = [\sigma']^{-1}(\lambda^2)$,

(6.8) $$v_1(\lambda) = -\int_0^{\varepsilon_1(\lambda)} \sqrt{\sigma'(s)}\, ds , \qquad \sqrt{\sigma'(\varepsilon_J)} < \lambda < \sqrt{\sigma'(0)} ,$$

(iii) $\varepsilon_1(\lambda) = \varepsilon_J$,

$$v_1(\lambda) = -\int_0^{\varepsilon_J} \sqrt{\sigma'(s)}\, ds \equiv v^*, \qquad \lambda_1 \equiv \sqrt{\frac{\varepsilon_\infty - \overline{\varepsilon}_J}{\varepsilon_\infty - \varepsilon_J}}\, c < \lambda < \sqrt{\sigma'(\varepsilon_J)} .$$

Behind the phase boundary, (4.7) leads to the exact expressions:

$$\varepsilon(x,t) = [(\varepsilon_J - v^*/c)\mathring{s}(t_b) + (-\sigma_J/c + v^* + D/c)] / [2(\mathring{s}(t_b) - c)]$$
$$+ [(\varepsilon_J + v^*/c)\mathring{s}(t_a) + (\sigma_J/c + v^* - D/c)] / [2(\mathring{s}(t_a) + c)] ,$$

(6.9) $$v(x,t) = [(\varepsilon_J - v^*/c)\mathring{s}(t_b) + (\sigma_J/c + v^* - D/c)] / [2(1 + \mathring{s}(t_b)/c)]$$
$$+ [(\varepsilon_J + v^*/c)\mathring{s}(t_a) + (\sigma_J/c + v^* - D/c)] / [2(1 + \mathring{s}(t_a)/c)] ,$$

where v^* is the same value defined in (6.8) and $t_a = t_a(x,t)$, $t_b = t_b(x,t)$ are, as before, depicted in Figure 4. Now it follows, either from Figure 4 — or by writing expressions for t_a and t_b — that $t_a \to \infty$, $t_b \to \infty$ in the limit $t \to \infty$, x/t fixed, whence $\mathring{s}(t_a) \to \mathring{s}(t_b) \to [(\varepsilon_\infty - \overline{\varepsilon}_J)/(\varepsilon_\infty - \varepsilon_J)]^{1/2} c$ in this same limit. Entering (6.9) with this result yields, as $t \to \infty$, $\lambda = x/t$ fixed,

$$\varepsilon(x,t) \to \varepsilon_\infty ,$$

(6.10)

$$v(x,t) \to -\sqrt{(\varepsilon_\infty - \varepsilon_J)/(\varepsilon_\infty - \bar{\varepsilon}_J)}\, c + v^* , \qquad 0 < \lambda < \lambda_1 .$$

For materials obeying (4.1) and values of ε_∞ subject to (6.6), the right-hand sides of (6.8), (6.10) furnish an exact solution of the form (2.4) to the problem $P(\infty)$ for the impulsive load

$$h(t) = \begin{cases} 0, & t = 0 , \\ \varepsilon_\infty > \bar{\varepsilon}_J, & t > 0 . \end{cases} \tag{6.11}$$

Since this solution also satisfies conditions (a) - (d) of section 2, we shall compare this solution to those impulsive load solutions obtained in that section.

For the materials now under consideration, condition (6.6) implies (and is implied by) Alt. 1 of (2.19). Recall that for this alternative, the impulsive load problem has a one parameter family of solutions $\bar{\bar{\varepsilon}}(\varepsilon^*,\lambda)$, $0 < \varepsilon^* < \varepsilon_J$. However, of these solutions, only $\bar{\bar{\varepsilon}}(\varepsilon_J,\lambda) = \tilde{\varepsilon}(\lambda)$ given by (2.17), (2.18), (2.22) has a region in front of the phase boundary with $\varepsilon = \varepsilon_J$. Indeed, it is easily verified that this $\tilde{\varepsilon}(\lambda)$ is given precisely by the right hand sides of (6.8), (6.10). Thus $\tilde{\varepsilon}(\lambda)$ given by (2.17), (2.18), (2.22) is — at least for materials obeying (4.1) and loads obeying (6.6) — appropriate for energy confining phase boundaries.

We close this paper on the remark concerning the case of a smooth $h(t) \to \varepsilon_\infty > \varepsilon_T$ for materials (4.1) with $\sqrt{\sigma'(\varepsilon_J)} < c$. Then, as shown, $\hat{t} < \infty$, and the solution ϕ of (4.19) does not lead to a solution of $P(T)$ for $T > \hat{t}$. Nevertheless, we conjectue that $P(\infty)$ has a solution involving an energy confining phase boundary and, moreover, anticipate that $\tilde{\varepsilon}(\lambda)$ given by (2.17), (2.18), (2.22) is the $\lambda = x/t$ fixed, $t \to \infty$ limit of the strain field. In particular, this predicts, in place of (6.4), that $\lim_{t\to\infty} \dot{s}(t) < \{[\varepsilon_\infty - \bar{\varepsilon}_J] / [\varepsilon_\infty - \varepsilon_J]\}^{1/2} c$. The problem $P(T)$, $T > \hat{t}$ may be studied in the framework developed here by treating the system (4.8), (4.9) which describes an energy confining phase boundary advancing into a more general strain and velocity field.

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Certain elastic solids when subjected to sufficiently high loads abruptly change their mechanical properties and yet continue to respond elastically to further loading. In one dimension such mechanically induced elastic phase transitions may be due to a non-monotonic stress-strain relationship. This appears to be particularly true for certain mineral crystals, such as calcite.

This work considers a one-dimensional dynamical problem for a special material. The problem reduces to determining the location of the internal moving boundary separating distinct elastic phases. This phase boundary is similar to

20. a gas dynamical shock wave. For the problem considered here, this phase boundary is shown to be governed by a functional equation of the form

$$\phi(\phi(t)) + F(\phi(t)) + t = 0$$

for the unknown $\phi(t)$, whose $F(\ )$ is a known function involving the boundary conditions. The unusual equation is derived by considering the effect of acoustic waves repeatedly reflecting between the phase boundary and the external boundary. The equation is shown to possess a unique solution and is treated asymptotically to determine the large-time behavior of the phase boundary.

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